Variations on Seymour's Conjecture

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Abstract

Seymour's Conjecture states that in any oriented simple graph, at least one vertex has "second-outdegree" at least as large as its outdegree. We investigate families of oriented graphs to see if they satisfy Seymour's Conjecture and to see if stronger claims can be made about the specific graph families. We verify the conjecture for the following families of digraphs:

(1) Acyclic digraphs
(2) k-out-regular oriented graphs where $k \leq 4$
(3) Cartesian products G□H in which the factors G and H satisfy the conjecture
(4) Certain kinds of Cayley graphs
(5) Out-regular bipartite graphs
(6) Pinwheel graphs PWC_n

In cases (2) [with k = 1,2], (4), (5) and (6), we prove a stronger claim that every vertex satisfies the conjecture. Finally, we show that if Seymour's Conjecture holds true for all strong oriented graphs, then it holds true for all oriented graphs.

1. Introduction

Seymour's Conjecture states that in any antisymmetric and loopless directed graph, at least one vertex has the property that the second-outdegree of that vertex is greater than or equal to the first-outdegree of that vertex. In order to fully explain the conjecture, it is necessary to first define some terms. Most of our definitions are taken from West's Introduction to Graph Theory [3], and the reader should consult that reference for any terms not defined here.

A directed graph (also know as a *digraph*) is a graph in which every edge connecting the vertices has a direction associated with it. If the edge is pointing out of a vertex *x*, we call it an *outedge* of *x*. If the edge is pointing out of vertex *x* and into vertex *y*, we say that *y* is an *outneighbor* of *x*. The *first-outdegree* of a vertex (also known as simply "the outdegree") is the number of outedges from that vertex. The outdegree of a vertex *x* is denoted by $d^+(x)$. The group of all first outneighbors of *x* is called the *first-outneighborhood* and is denoted by $N^+(x)$. The *second-outdegree* of a vertex *x* is the number of vertices in the digraph that can be reached in 2 steps exactly, not including the vertices reachable in one step from *x*. This is denoted by $d^{++}(x)$.

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The *second-outneighborhood* of x is denoted $N^{++}(x)$ and includes all vertices reachable in two steps or less from x. We define the neighborhood of vertices reachable in two steps but not one from x as the *new-second-outneighborhood* and we denote in by $NN^{++}(x)$. The *indegree* of a vertex is the number of edges coming into it and is denoted by $d^{-}(x)$. A digraph is called *antisymmetric*, if whenever y is an outneighbor of x, x is not an outneighbor of y. A graph is loopless if no vertex is its own outneighbor, i.e. no outedge is an in-edge of the same vertex. Another name for an antisymmetric and loopless digraph is an *oriented graph*. In this paper, whenever the term *digraph* is used, it will be assumed that it is antisymmetric and loopless. Thus, with our denotations, Seymour's Conjecture would read "In any digraph, there is at least one vertex, say x, for which $d^{+}(x) \leq d^{++}(x)$."

The above statement of Seymour's Conjecture is not the only way that it can be stated. We can look at the *square* of a digraph, denoted D^2. If we take a digraph D, then the square of D is a new digraph in which we draw a directed edge from a vertex to all the vertices that it can reach in 2 or fewer steps in D (see figure 1.1). Using this terminology, Seymour's Conjecture is "For any digraph D, there is at least one vertex x, for which $d^{+}(x)$ in D is less than or equal to $2d^{+}(x)$ in D^2."

D $\qquad$ D^2

Figure 1.1

Seymour's Conjecture has been proven for a particular family of oriented graphs known as tournaments. A tournament is a digraph in which for every two vertices x and y, exactly one of the two directed edges xy or yx is present. (Thus a tournament might represent the outcomes of who defeated who in a round-robin sports tournament in which there are no ties, and each pair of participants compete head to head exactly once.) Seymour's Conjecture applied to tournaments is now known as Dean's Conjecture and was first proved by David Fisher [1] and later proved in a shorter way by Frédéric Havet and Stéphan Thomassé [2]. Havet and Thomassé went on to prove the stronger claim that a tournament with no dominated vertex (outdegree of 0) has at least two vertices that satisfy Seymour's Conjecture.

In this paper we study specific families of digraphs and prove that such graphs do satisfy Seymour's Conjecture. We also make stronger claims for some of these digraphs, specifically the claim that in some of the digraphs we studied, *all* of the vertices have a second-outdegree greater than or equal to their first-outdegree. We end our paper by showing that if Seymour's Conjecture can be proved for all strong digraphs then it follows that it holds for all reducible digraphs as well.

2. Acyclic Digraphs

Definition 2.1: An *acyclic digraph* is a directed graph containing no walk which starts and ends at the same vertex, but otherwise repeats no vertices.

Lemma 2.2: *In any acyclic digraph, there exists a vertex with outdegree 0.*

Proof: Let A be an acyclic digraph. We are done unless every vertex in A has positive outdegree, that is, $\delta^+(A) > 0$. Consider a longest path $x_1, x_2, ..., x_n$ in *A*, where x_n is the last vertex of the path. Assuming that every vertex has positive outdegree gives us two cases for x_n.

Case 1: Vertex x_n has an outedge to at least one of the other vertices in the path, $x_1, x_2, ..., x_{n-1}$, but this is a contradiction since *A* is acyclic.

Case 2: Vertex x_n has an outedge to a new vertex, x_{n+1}, but this is a contradiction since the path considered was longest.

Thus, x_n must have outdegree 0, as was to be shown.

Theorem 2.3*: If $d^+(x) = 0$ for some x in a digraph D, in other words, $\delta^+(D) = 0$, then Seymour's Conjecture holds for D. In particular, all acyclic digraphs satisfy Seymour's Conjecture.*

Proof: Let *D* be an acyclic digraph. By Lemma 2.2, we know that there exists a vertex *x* in *D* such that $d^+(x) = 0$, so it suffices to prove the first part of the theorem. Let *x* be a vertex of outdegree zero. It is clearly the case that the second-outdegree of *x* is at least 0 since the outdegree of a vertex is never negative. So, $d^+(x) = 0 \le d^{++}(x)$ which satisfies Seymour's Conjecture.

3. Cayley Graphs with Additive Generators

3.1 The Infinite Case

Definition 3.1.1: The *infinite Cayley line with additive generators* is a directed graph determined by a set of generators $\Gamma = \{g_1, g_2, ..., g_k\}$, where $g_i \in$ **Z** and $0 \notin \Gamma$. The vertices of the graph correspond to the elements of the integers, and whenever $g_i + a = b$, (a,b∈ **Z**) an edge is drawn from a to b.

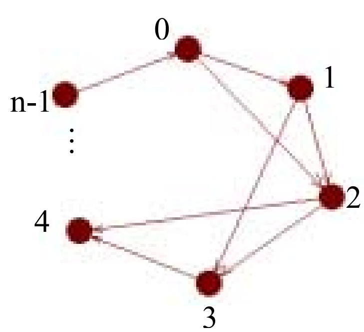
Theorem 3.1.2: *All vertices in the infinite Cayley line with additive generators satisfy Seymour's Conjecture.*

Proof: Let the additive generators be $\Gamma = \{g_1, g_2, ..., g_k\}$, where $g_i \in$ **Z**, $g_1 < g_2 < ... < g_k$, and $0 \notin \Gamma$. It is clear that the infinite Cayley line is vertex-transitive, so investigating one vertex is as good as any other. Therefore, it suffices to consider the vertex 0 and show that $d^+(0) \le d^{++}(0)$. We know that the first-outneighborhood of 0 is the generator set, that is, $N^+(0) = \Gamma = \{g_1, g_2, ..., g_k\}$. The second-outneighborhood of 0 is generated by adding each member of the generator set to each member of the first-outneighborhood of 0. To obtain new second-outneighbors, create a division within the generators in order to separate the negative generators from the positive generators, so that $\Gamma = \{g_{N_1}, g_{N_2} ..., g_{N_m}\} \cup \{g_{P_1}, g_{P_2}, ..., g_{P_n}\}$, where $g_{N_1}, g_{N_2}, ..., g_{N_m} < 0$ and $g_{P_1}, g_{P_2}, ..., g_{P_n} > 0$. Now, consider the most negative first-outneighbor of 0, denoted g_{N_1}. We are guaranteed that $g_{N_1} + g_{N_i}$ (where $1 \le i \le m$) is a new second-outneighbor of 0 since $g_{N_1} + g_{N_i} < g_{N_1}$, where g_{N_1} is the most negative first-outneighbor of 0. Likewise, consider the most positive first-outneighbor of 0, denoted g_{P_n}. We are guaranteed that $g_{P_n} + g_{P_j}$ (where $1 \le j \le n$) is a new second-outneighbor of 0 since $g_{P_n} + g_{P_j} > g_{P_n}$, where g_{P_n} is the most positive first-outneighbor of 0. To summarize, we know that we have found at least m new negative second-outneighbors and n new positive second-outneighbors. Since $m + n = |\Gamma|$, $d^+(0) = |\Gamma| \le d^{++}(0)$ which satisfies Seymour's Conjecture as was to be shown.

3.2 The Modulo n Case

The Cayley graph modulo n case works very similarly to the infinite case. For a subset Γ of V = {0,1,2,…,n-1}, we define the Cayley graph C(Γ, n) as being the digraph with vertex set V, in which there is an edge from x to x + e iff e ∈ Γ. The Cayley graph modulo n is a special digraph that is better understood if pictured in a clock or circle form.

Here is an example of a Cayley graph of modulo n with Γ = {1, 2}



We consider only what we call a Restricted Cayley graph (RCG) in which the following hold true:

- $0 < e < n/2$ for each e ∈ Γ.
- Symmetric edges or bi-edges are not allowed.
  - Inverses are prohibited.
    - If e ∈ Γ, then e ∉ Γ.
    - If e ∈ Γ, then (n-e) ∉ Γ.

Lemma 3.2.1: *For every vertex x, $d^+(x) = |\Gamma|$.*

Proof: Let $|\Gamma| = k$. Refer to the definition of the Cayley graph previously mentioned, and note that there will be k directed edges coming from each vertex. Therefore, for every x, $d^+(x)$ is the number of outedges of x, which is $k = |\Gamma|$.

Theorem 3.2.2: *If $|\Gamma| > 1$, then for D, any RCG and any vertex x in V, and if Γ not in the form of $\{\ell, 2\ell, 3\ell,\ldots, k\ell\}$ for some $\ell \in Z$, then $d^{++}(x) \geq |\Gamma| + 1$.*

Proof: Assume that the elements of Γ are listed in ascending order. Choose an arbitrary vertex x in the digraph. Based on Lemma 3.2.1, we can assume that this vertex has k directed edges that go to k different vertices, and each one of those vertices has k directed outedges themselves. Similar to the proof of infinite case, by vertex transitivity, it suffice to consider x = 0.

Case 1: Suppose Γ consists of $\{\ell, 2\ell, 3\ell,\ldots, k\ell\}$.
Notice that Γ consists of consecutive multiples of ℓ. Consider adding each element in Γ to every other element in Γ including itself to generate the list of elements in $N^{++}(0)$. Similarly in the infinite case, elements located the in $N^{++}(0)$ that repeat elements in Γ will not be considered as an element in $NN^{++}(0)$ and must be thrown out. Elements located in $N^{++}(0)$ yet not in $\Gamma = N^+(0)$, will be counted exactly once even if they "repeat". Here, $N^{++}(0) = \{e_i + e_j \mid e_i, e_j \in \Gamma\} = \{2\ell, 3\ell,\ldots, 2k\ell\}$. Thus $N^{++}(0) \setminus N^+(0) = NN^{++}(0) = \{(k+1)\ell, \ldots, 2k\ell\}$. So $d^{++}(x), = k = |\Gamma|$, which shows that Seymour Conjecture holds for this particular case.

Case 2: Γ consists of any other set of k elements.
The elements of $N^{++}(0)$ are as follows:

$\underline{e_1 + e_1}$, $e_1 + e_2$, $e_1 + e_3$, …, $e_1 + e_k$
$e_2 + e_1$, $\underline{e_2 + e_2}$, $e_2 + e_3$, …, $e_2 + e_k$
$e_3 + e_1$, $e_3 + e_2$, $\underline{e_3 + e_3}$, …, $e_3 + e_k$
⋮
$e_k + e_1$, $e_k + e_2$, $e_k + e_3$, …, $\underline{e_k + e_k}$

Notice that all the elements located below the diagonal are equal to corresponding elements above the diagonal. Without loss of generality, let's just assume that the elements below the diagonal will be thrown out, thereby regarding $\frac{k^2 - k}{2}$. This leaves $\frac{k^2 + k}{2}$ elements remaining to be considered in the count for $NN^{++}(0)$. Note that the elements $(y + e_k)$, where $y \in [e_1, e_k]$, will consist of k new elements. Thus, $|NN^{++}(0)| \geq k$. This basically shows that Seymour's Conjecture holds for this particular digraph D simply because $d^{++}(0) \geq d^+(0) = k$.

For a stronger case, consider adding the smallest element, e_1, to each element in $[e_1, e_{k-1}]$. It suffice to know that at least one such sum that will not repeat an element in Γ or from $e_i + e_k$. This is simply because $e_1 \leq e_j$ and $e_i < e_k$, and therefore $e_1 + e_i \neq e_j + e_k$, where $i, j \in [e_1, e_{k-1}]$.

Case 2.1: Suppose Γ contains (e_1, $2e_1$, $3e_1$,…, me_1), and $me_1 = e_k$. Also, Γ contains some non-multiple of e_1. Choose the largest e_i element that's not a multiple of e_1, and add it to e_1. You will produce a new element that's not in Γ. This is because regardless of what e_i is, the element (e_1 + e_i) will not be a multiple of e_1, and thus will not equal me_1.

Case 2.2: Suppose Γ contains (e_1, $2e_1$, $3e_1$,…, me_1) and Me_1 for some M > (m + 1). Also, (m+1) $\notin$ Γ. Adding the element e_1 to me_1 will give you (m+1)e_1, a new element that's not in Γ.

Case 2.3: Suppose Γ contains (e_1, $2e_1$, $3e_1$,…, me_1), and e_k which is not a multiple of e_1. Since me_1 is the last multiple in Γ, adding the element e_1 to me_1 will give you (m+1)e_1, a new element that's not in Γ.

We conclude that the $d^{++}(x) \geq k + 1 \geq |\Gamma| + 1 \geq d^{+}(x) + 1$. Thus, Seymour Conjuncture holds for the modulo n family of Cayley graphs with the noted restrictions.

4. Out-Regular Digraphs

An *out-regular digraph* is a digraph in which every vertex has the same first-outdegree. In this section, we will prove that Seymour's Conjecture holds for k-out-regulars with k≤4, and we will conjecture that it holds for all out-regular digraphs.

Lemma 4.1: *In any 1-out-regular digraph, every vertex has a second-outdegree equal to its first-outdegree.*

Proof: Let D be a 1-out-regular digraph. Assume an arbitrary vertex, say *x*, defeats *y*. Because D is antisymmetric and loopless, we know that the edge exiting *y* must defeat a new vertex, say *z*. We know that *y* is the only first-outneighbor of *x* and that z is the only first-outneighbor of *y*. Thus, $d^{+}(x)=1$ and $d^{++}(x)=1$ for any vertex in D.

Lemma 4.2: *In any 2-out-regular digraph, every vertex has second-outdegree greater than or equal to its first-outdegree and thus satisfies Seymour's Conjecture.*

Proof: Let D be a finite 2-out-regular digraph. Assume that an arbitrary vertex, say *x*, defeats two vertices *y* and *z*. Neither *y* nor *z* can defeat *x* because D is antisymmetric. It is only possible for one edge to connect *y* and *z*. Without loss of generality, we can assume that *z* does not defeat *y* (since *y* and *z* cannot both defeat each other). Then *z* must have two outneighbors other than *x* and *y* thus $d^{++}(x) \geq 2 = d^{+}(x)$. This is true for every vertex in D.

Lemma 4.3: *In any 3-out-regular digraph, at least one vertex will satisfy Seymour's Conjecture.*

Proof: Consider an arbitrary vertex, say *w*, in a 3-out-regular digraph D. Consider $N^{+}(w)$, which say consists of vertices *x*, *y*, *z*. Let S be the subgraph induced by these vertices. We show that at least one of *w*, *x*, *y*, *z* satisfies Seymour's Conjecture.

Case 1: At least one of x, y, z has an internal outdegree within S of 0. Without loss of generality, let it be x. We know that x cannot have an edge back to w, so since x defeats neither y nor z, then x must have 3 new-outneighbors, i.e. 3 new-second-outneighbors of w, and thus w satisfies Seymour's Conjecture.

Case 2: At least one of x, y, z has an internal outdegree of exactly 1 within S, and none have internal outdegree of 0 (because then we would be dealing with case 1). On 3 vertices, there can be at most 3 edges, thus since none of the vertices have internal outdegree of 0, we can say that the internal outdegree of each of x, y, z is exactly 1.

Since the internal outdegree of each of the vertices of S is 1, we know that there are 2 new-outneighbors of each (that are not in S and not w). If the new-outedges of x, y, z do not all defeat the same 2 new vertices, then there are 3 or more ways out of S, making $d^{++}(w) \geq 3 = d^{+}(w)$, and hence w will satisfy Seymour's Conjecture. Thus, let's assume that all of the edges out of S do go to the same two vertices, which we'll call a and b. Since there are 2 new-outneighbors of each of x, y, z, we know that both a and b are defeated by all of S, hence there can be no return edges from a or b to S. Also, we know that at most there can be one edge directly connecting a and b. Without loss of generality, let's say that b does not defeat a. Hence, b will have 3 new-outneighbors (not in S and not a), giving each of x, y, z a second-outdegree of at least 3. Hence, each of x, y, z will satisfy Seymour's Conjecture.

Case 3: At least one of x, y, z has internal outdegree within S of 2. If this is the case then we know that at least one other vertex in S has internal outdegree of 0, and this case is thus proved by case 1.

Lemma 4.4: *Let D be a 4-out-regular digraph. At least one vertex in D satisfies Seymour's Conjecture.*

Proof: Consider an arbitrary vertex v in a 4-out-regular digraph and consider $N^{+}(v)$, which say consists of vertices w, x, y and z. We show that at least one of v, w, x, y, z satisfies Seymour's Conjecture.

Case 1: If any of w, x, y, or z do not defeat at least one other vertex in $D[N^{+}(v)]$ (the subgraph induced by w, x, y and z), then that vertex will have 4 new-outneighbors (not included in $D[N^{+}(v)]$), i.e. 4 new-second-outneighbors of v, and thus v will satisfy Seymour's Conjecture.

Case 2: Each vertex w, x, y, z defeats at least one other vertex within $D[N^{+}(v)]$. By argument of averaging, we know that at least one of the vertices in $D[N^{+}(v)]$ must have internal outdegree at most 1, as follows:

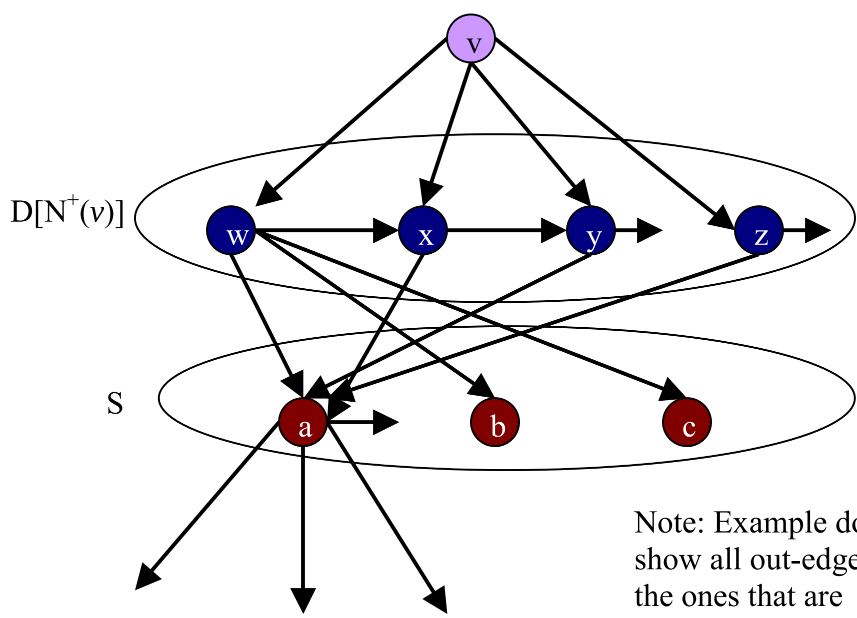
> (Averaging argument: The most edges possible between 4 vertices is 4 choose 2, which equals 6. The average outdegree in $D[N^{+}(v)]$ is the sum of the edges divided by the number of vertices in $D[N^{+}(v)]$. The sum of the edges is at most 6 and the number of vertices is always 4. Hence, the average outdegree in $D[N^{+}(v)]$ is always $\leq 6/4 = 1.5$. We know that for the average to be no higher than 1.5, at least one of the vertices must have outdegree less than or equal to the average. Thus, since the average outdegree is 1.5 and

each vertex has outdegree at least 1, we know that at least one vertex in D[N$^+$(v)], say w, must have outdegree of 1 within D[N$^+$(v)].)

We know that w has 3 new-outneighbors a, b and c, outside of D[N$^+$(v)]. If any of the vertices in D[N$^+$(v)] has an outneighbor other than w, x, y, z, a, b, or c, then we know that there are at least 4 second-outneighbors of v not included in D[N$^+$(v)] and thus that v satisfies Seymour's conjecture. Hence, let's assume that every edge out of D[N$^+$(v)] goes to either a, b, or c, and let S denote the subgraph induced by a, b, and c. We know that at least one of a, b and c is defeated by all of the vertices w, x, y and z, because at most, there are 6 internal edges in D[N$^+$(v)] and we know that we are dealing with 16 outedges total from the 4 vertices in D[N$^+$(v)]. This leaves us with at least 10 edges coming out of D[N$^+$(v)] and all going to a, b, and c. Thus, with at least 10 edges going into 3 vertices, at least one of those vertices will have 4 of those edges going into it (which would be from the 4 vertices w, x, y and z). By this same argument, we also know that each of a, b and c has at least 2 in-edges from D[N$^+$(v)]. Without loss of generality, let's assume that everything in D[N$^+$(v)] defeats vertex a. This tells us that a can have no edges returning to D[N$^+$(v)]. There are 3 cases that can occur at vertex a:

Case 2.1: The outdegree of a within S is 0. If this is the case, then we know that a will defeat 4 new vertices not in S or D[N$^+$(v)] and hence that w, x, y and z all satisfy Seymour's Conjecture.

Case 2.2: The outdegree of a within S is 1. If this is the case then we know that a will defeat 3 new vertices not in S or D[N$^+$(v)]. Now let's look at the vertex w in D[N$^+$(v)] with internal outdegree equal to one. We know that the vertex that w defeats within D[N$^+$(v)], let's say x, will also have an internal outdegree of at least one, so without loss of generality x defeats y. We also know that x will not defeat w or itself. Hence, w has at least the new-second-outneighbor y within D[N$^+$(v)] and 3 new-second-outneighbors outside of both S and D[N$^+$(v)] (the outneighbors of a). Hence, $d^{++}(w) \geq 4$ and w satisfies Seymour's Conjecture.



Note: Example does not show all out-edges, just the ones that are mentioned in *Case2.2*.

Case 2.3: The outdegree of a within S is 2. We already defined a as being defeated by all of w, x, y and z. This leaves at least 6 other edges coming out of D[N$^+$(v)] and going into S. We also know that within the subgraph S, there can be at most 3 internal edges. In this case, there are two things that could happen:

> *Case 2.3.1*: Either b or c has exactly 2 edges coming into it from D[N$^+$(v)] (without loss of generality, let's say b). This means that c must be defeated by all of w, x, y and z. We also know that c has internal outdegree in S of either 0 or 1 because two edges are already coming out of a. Because c is defeated by all of w, x, y and z, it cannot have any edges going back to that neighborhood. Thus, if the internal outdegree of c in D[N$^+$(v)] is 0 then w, x, y and z will all satisfy Seymour's conjecture (proved in case 2.1). If the internal outdegree of c is one, then vertex w will satisfy Seymour's Conjecture (proved in case 2.2).
>
> *Case 2.3.2*: There are at least 3 edges into each of b and c from D[N$^+$(v)]. Because of this, only one edge could possibly return from each of b and c to D[N$^+$(v)]. We know that the internal outdegree of either b or c will be 0 (without loss of generality, let's say b) and thus that b will have at least 3 new-outneighbors outside of both D[N$^+$(v)] and S. Hence, considering vertex w in D[N$^+$(v)] with new-second-outneighbor within D[N$^+$(v)], we know that the second-outdegree of w will be at least 4 and that w will satisfy Seymour's Conjecture.

Conjecture 4.5: *At least one vertex in all out-regular digraphs will satisfy Seymour's Conjecture.*

We believe that similar types of arguments as used for the 3-out-regular and 4-out-regular digraphs would work, although as the out-degree gets higher, the argument will predictably have more cases.

5. k-Out-Regular Bipartite Graphs

Definition 5.1: An *oriented bipartite graph* is a directed graph whose vertex set can be partitioned into two disjoint, independent sets called partite sets.

Theorem 5.2: *All vertices of every k-out-regular oriented bipartite graph satisfy Seymour's Conjecture.*

Proof: Let P be any k-out-regular oriented bipartite graph with partite sets A and B. Consider any vertex x where, without loss of generality, $x \in$ A. Vertex x must have k first-outneighbors as P is k-out-regular, and each of these first-outneighbors must be in B as P is bipartite. Consider any first-outneighbor y of x, where $y \in B$. Vertex y must also have k first-outneighbors, and each is a second-outneighbor of x. We know that these first-outneighbors of y are <u>new</u>-second-outneighbors of x since there can be no edge from y to any element of B by the definition of bipartite and there can be no edge from y to x since there is already and edge from x to y. Thus, $d^+(x) = k \leq d^{++}(x)$ which satisfies Seymour's Conjecture as was to be shown.

6. PWC_n: A Pinwheel Graph

Definition 6.1: K_n denotes the complete graph on vertex set $[n] = \{1,2,\ldots,n\}$. A *triplet* in $L(K_n)$, the line graph of K_n, is the subgraph induced by three vertices, ab, ac, and bc where a,b,c are chosen from [n].

Definition 6.2: PWC_n denotes a pinwheel graph which is a directed line graph of the clique on n elements where every triplet forms a directed three-cycle.

Theorem 6.3: *All vertices in any PWC_n satisfy Seymour's Conjecture.*

Proof: Consider any arbitrary vertex ab in a PWC_n. We know that vertex ab is an element of $n-2$ triplets by a property of PWC_n, where every triplet contains exactly one outneighbor of ab. So, $d^+(ab) = n-2$. Consider any first-outneighbor ac of vertex ab. We know that the third vertex in the triplet containing both ab and ac is bc which is a first-outneighbor of ac since the triplet's edges are oriented cyclically. Thus, bc is a second-outneighbor of ab. Also, it is not hard to see that bc appears only once in all of the triplets containing ab since triplets share only one vertex, and ab is that vertex. Therefore, $d^+(ab) = n-2 \leq d^{++}(ab)$ as was to be shown.

7. Cartesian Products

Definition 7.1: The *Cartesian product* of digraphs G and H, written G□H, is the graph with vertex set V(G)×V(H) specified by putting an edge from (u, v) to (u',v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$ [3].

G and H are known as the *factor graphs* of the Cartesian product. The number of vertices in the Cartesian product is the number of vertices in G times the number of vertices in H. (See figure 7.2 for an example.)

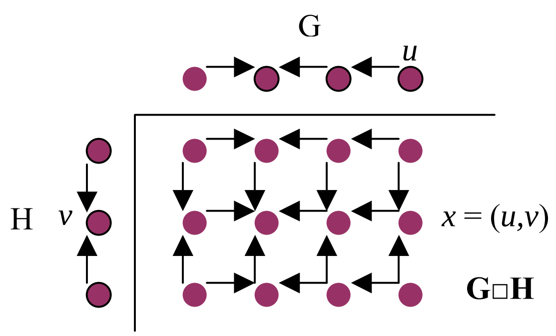


Figure 7.2

Theorem 7.3: *If both factor graphs of a Cartesian product each have at least one vertex that satisfies Seymour's Conjecture, then there will be at least one vertex in the Cartesian product which satisfies Seymour's Conjecture.*

Proof: Let G□H be the Cartesian product of two directed graphs G and H. Assume that G and H each have at least one vertex for which the second-outdegree is greater than or equal to the first-outdegree, say vertex *u* in G and vertex *v* in H. (Refer to figure 7.2.)

Consider vertex (u,v) of G□H and denote that vertex *x*. We know that in G□H, $d^+(x) = d_G^+(u) + d_H^+(v)$. This is because *x* has as its outneighbors those corresponding to *v* in column *u* and the outneighbors corresponding to *u* in row *v*. No other edges than these come out of *x* because in a Cartesian product the edges are only defined on the rows and columns. As *x* has at least all the new-second-outneighbors corresponding to *u* in row *v* and all the new-second-outneighbors corresponding to *v* column *u*, it follows that in G□H, $d^{++}(x) \geq d_G^{++}(u) + d_H^{++}(v)$. Hence, because we know that $d_G^+(u) \leq d_G^{++}(u)$ and that $d_H^+(v) \leq d_H^{++}(v)$, it follows that $d_G^+(u) + d_H^+(v) \leq d_G^{++}(u) + d_H^{++}(v)$. Therefore, $\mathbf{d^+(x)} = d_G^+(u) + d_H^+(v) \leq d_G^{++}(u) + d_H^{++}(v) \leq \mathbf{d^{++}(x)}$, giving us $\mathbf{d^+(x) \leq d^{++}(x)}$, which tells us that at least vertex *x* in G□H satisfies Seymour's conjecture.

8. Seymour's Conjecture and Strong Digraphs

Theorem 8.1: *If Seymour Conjecture holds true for strong oriented graphs, then it holds true for all oriented graphs.*

Proof: Assume that Seymour Conjecture holds true for all strong oriented digraphs. A digraph is strong when you are able to get from any vertex to all other vertices. Consider a non-strong or reducible digraph D. By definition of reducible, D can be broken into smaller strong components (sc's), in which there exists a transitive relationship, that is every vertex in the ith component of sc will have an outedge directly connected to the jth component of sc, where i > j. This is usually represented with a downward arrow (⇓). See figure 8.2.

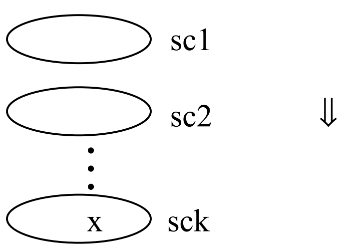


Figure 8.2

Choose an arbitrary vertex x in sck. The outdegree of x in sck would be the same as the outdegree of x in D. This is simply because sck is a subgraph of D and there is no edges coming out of sck. Since Seymour's Conjecture is assumed to hold for strong digraphs, then twice the outdegree of x in the subgraph sck is less than or equal to the outdegree of x in the square subgraph of sck. That is, $d_D^{++}(x) = d_{sck}^{++}(x) \geq d_{sck}^{+}(x) = d_D^{+}(x)$. Therefore, $d_D^{++}(x) \geq d_D^{+}(x)$, meaning there exist at least one vertex in D for which Seymour's Conjecture holds.

9. Conclusion

We have shown that Seymour's Conjecture holds true for the different cases described previously. Our strongest aspect is Theorem 8.1, which states that if Seymour's Conjecture holds true for strong oriented graphs, then it holds true for all oriented graphs. This is indeed a powerful theorem simply because if it is proven that the conjecture works for strong digraphs, then the conjecture would be proved, and thus become an important theorem in graph theory.

Although easy to state and understand, Seymour's Conjecture leaves many open questions. There are still many different cases and graphs that we would like to investigate. Among those are near-tournaments, (in which you take a tournament and remove an edge), a stronger claim for the k-out regular digraphs as well as the Cayley graph on modulo n, and of course the proof of the conjecture holding for all strong digraphs.

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